Theorem 1.

$$\frac{\partial g}{\partial g_{\mu\nu}} = g^{\mu\nu} g \tag{1}$$

or in terms of matrix elements

$$\frac{\partial \det(\mathbf{g})}{\partial \mathbf{g}_{\bar{\mu}\bar{\nu}}} = (\mathbf{g}^{-1})_{\bar{\mu}\bar{\nu}} \det(\mathbf{g}) \tag{2}$$

Proof. Starting with $(\delta\mathbf{g})_{\bar{\mu}\bar{\nu}} = \Delta$ and 0 for other elements,

$$\begin{aligned}
\frac{\partial \det(\mathbf{g})}{\partial \mathbf{g}_{\bar{\mu}\bar{\nu}}} &= \lim_{\Delta\to 0} \frac{1}{\Delta} \det(\mathbf{g} + \delta\mathbf{g}) - \det(\mathbf{g}) = \\
&= \lim_{\Delta\to 0} \frac{1}{\Delta} \det(\mathbf{g}) \det(1 + g^{-1}\delta\mathbf{g}) - \det(\mathbf{g}) = \\
&= \lim_{\Delta\to 0} \frac{1}{\Delta} \det(\mathbf{g})(1 + \mathrm{Tr}(g^{-1}\delta\mathbf{g}) + O(\Delta^2)) - \det(\mathbf{g}) = \\
&= \det(\mathbf{g}) \lim_{\Delta\to 0} \frac{1}{\Delta} \mathrm{Tr}(\mathbf{g}^{-1}\delta\mathbf{g}) = \det(\mathbf{g}) \lim_{\Delta\to 0} \frac{1}{\Delta} (\mathbf{g}^{-1})_{ji} (\delta\mathbf{g})_{ij}
\end{aligned}$$

Note for every summed index the expression is 0, expect $ij = \bar{\mu}\bar{\nu}$. Therefore,

$$\frac{\partial \det(\mathbf{g})}{\partial \mathbf{g}_{\bar{\mu}\bar{\nu}}} = \det(\mathbf{g}) \lim_{\Delta\to 0} \frac{1}{\Delta} (\mathbf{g}^{-1})_{\bar{\nu}\bar{\mu}} \Delta = \det(\mathbf{g})(\mathbf{g}^{-1})_{\bar{\nu}\bar{\mu}}$$

As both g and its inverse are symmetric, this concludes the proof. □

Theorem 2.

$$\frac{\partial g}{\partial g^{\mu\nu}} = -g_{\mu\nu} g \tag{3}$$

Proof. By using the previous result, and chain rule, one gets

$$\begin{aligned}
\frac{\partial \det(\mathbf{g})}{\partial (\mathbf{g}^{-1})_{\bar{\mu}\bar{\nu}}} &= \frac{\partial (\det(\mathbf{g}^{-1}))^{-1}}{\partial (\mathbf{g}^{-1})_{\bar{\mu}\bar{\nu}}} = -\frac{1}{\det(\mathbf{g}^{-1})^2} \frac{\partial \det(\mathbf{g}^{-1})}{\partial (\mathbf{g}^{-1})_{\bar{\mu}\bar{\nu}}} = \\
&= -\frac{1}{\det(\mathbf{g}^{-1})^2} \det(\mathbf{g}^{-1})(\mathbf{g})_{\bar{\nu}\bar{\mu}} = -\det(\mathbf{g})(\mathbf{g})_{\bar{\nu}\bar{\mu}}
\end{aligned}$$

□

Theorem 3.

$$\frac{\partial g^{\alpha\beta}}{\partial g_{\mu\nu}} = -g^{\alpha\mu}g^{\nu\beta} \tag{4}$$

Proof. Starting from $g_{\sigma\alpha}g^{\alpha\beta} = \delta^{\gamma}_{\sigma}$ one derives

$$\begin{aligned}
\frac{\partial}{\partial g_{\mu\nu}}(g_{\sigma\alpha}g^{\alpha\beta}) &= 0 \\
\frac{\partial g_{\sigma\alpha}}{\partial g_{\mu\nu}}g^{\alpha\beta} + g_{\sigma\alpha}\frac{\partial g^{\alpha\beta}}{\partial g_{\mu\nu}} &= 0 \\
\delta^{\mu}_{\sigma}\delta^{\nu}_{\alpha}g^{\alpha\beta} + g_{\sigma\alpha}\frac{\partial g^{\alpha\beta}}{\partial g_{\mu\nu}} &= 0 \\
\delta^{\mu}_{\sigma}g^{\nu\beta} + g_{\sigma\alpha}\frac{\partial g^{\alpha\beta}}{\partial g_{\mu\nu}} &= 0 \\
g^{\alpha\sigma}\delta^{\mu}_{\sigma}\delta^{\nu}_{\alpha}g^{\alpha\beta} + \frac{\partial g^{\alpha\beta}}{\partial g_{\mu\nu}} &= 0 \\
g^{\alpha\mu}g^{\nu\beta} + \frac{\partial g^{\alpha\beta}}{\partial g_{\mu\nu}} &= 0
\end{aligned}$$

Which gives

$$\frac{\partial g^{\alpha\beta}}{\partial g_{\mu\nu}} = -g^{\alpha\mu}g^{\nu\beta}$$

□

Theorem 4.

$$T^{\mu\nu} = \frac{2}{\sqrt{g}}\frac{\partial S}{\partial g_{\mu\nu}} \iff T_{\mu\nu} = -\frac{2}{\sqrt{g}}\frac{\partial S}{\partial g^{\mu\nu}} \tag{5}$$

Proof.

$$T^{\mu\nu} \overset{(?)}{=} \frac{2}{\sqrt{g}}\frac{\partial S}{\partial g_{\mu\nu}} = \frac{2}{\sqrt{g}}\frac{\partial g^{\alpha\beta}}{\partial g_{\mu\nu}}\frac{\partial S}{\partial g^{\alpha\beta}} = -\frac{\partial g^{\alpha\beta}}{\partial g_{\mu\nu}}T_{\alpha\beta} = g^{\mu\alpha}g^{\nu\beta}T_{\alpha\beta}$$

□